

Non-Cooperative Quantum Game Theory

Chiu Fan Lee* and Neil F. Johnson†

Centre for Quantum Computation and Physics Department
Clarendon Laboratory, Oxford University
Parks Road, Oxford OX1 3PU, U.K.

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Abstract

The physical world obeys the rules of quantum, as opposed to classical, physics. Since the playing of any particular game requires physical resources, the question arises as to how Game Theory itself would change if it were extended into the quantum domain. Here we provide a general formalism for *quantum* games, and illustrate the explicit application of this new formalism to a quantized version of the well-known prisoner's dilemma game.

Key words: quantum games, non-cooperative games, the prisoner's game

*c.lee1@physics.ox.ac.uk

†n.johnson@physics.ox.ac.uk

1 Introduction

Quantum mechanics revolutionized physics a century ago. During the past decade, quantum mechanics has extended its impact into the fields of information theory and computer science. By manipulating qubits—where ‘qubit’ denotes the quantum equivalent of the single bit from classical information theory—researchers have made some remarkable discoveries: perfectly secure cryptography is feasible [4, 10, 3], certain computational tasks can be performed more efficiently than their classical counterparts [22, 12], and teleportation and superdense coding have been proved possible [5, 6]. The key common element in each of these applications is information—and information is ultimately a physical quantity since it needs to be stored on, and manipulated by, a physical system. Games are no different in that their physical implementation (i.e. the playing of the game) will also require a physical system. In particular, the actions of players in games can ultimately be broken down into yes/no responses to a series of specific questions posed by an external referee. Since such a yes/no binary response has a natural quantum equivalent in terms of a qubit, game theory becomes an obvious candidate for the incorporation of quantum mechanical effects.

The study of quantum games started in 1999 [20]. Eisert, Wilkens and Lewenstein later proposed a quantized version of the prisoner’s dilemma game, claiming that the resulting ‘quantum game’ resolves the prisoner’s dilemma [9]. This particular conclusion was later criticized by Benjamin and Hayden on the grounds that the quantum strategies considered were limited in an unphysical way [1]. However the theoretical framework introduced by Eisert and co-workers is unaffected by this criticism, and moreover underlies most of the subsequent research in quantum games [7, 2, 14, 15, 18, 16, 13, 11]. In this paper, we introduce a general formalism of quantum games based on Eisert *et al*’s framework, and study the quantized prisoner’s dilemma game as a specific example.

Before presenting a detailed discussion of quantum games, we start by motivating the general formalism. In particular, we will single out the essential elements in classical non-cooperative game theory and then argue how these elements motivate the introduction of the corresponding elements in the quantum version. We note that from now on, all games considered will be non-cooperative and finite. We will also restrict ourselves to two-player games, but will comment on how this can be generalized.

Classically, any game is fully described by its corresponding payoff matrix. For instance, the prisoner’s dilemma game may be represented

by the following payoff matrix which has two rows (labelled by 0 and 1, for example) and two columns:

$$\begin{bmatrix} (3, 3) & (0, 5) \\ (5, 0) & (1, 1) \end{bmatrix}. \quad (1)$$

We can then ignore the underlying motivation for the game in question, since playing the game becomes equivalent to picking a number corresponding to a particular row (or column for the second player) of the game matrix, e.g. 0 or 1. Indeed, one of the successes of game theory may be seen as the incorporation of a utility function into entries of a game matrix, hence making a mathematical treatment possible. After the players have made their choices, e.g. by writing 0 or 1 on separate pieces of paper, someone needs to collect together this information and distribute the corresponding payoffs. Hence we assume the existence of a referee whose sole purpose is to collate the choices made by the players and to assign the corresponding payoffs. In short, playing a game constitutes an exchange of information between the players and the referee. The messages exchanged can generally be thought to be encoded as bit-strings of fixed length, hence strengthening the information-theoretic theme. In the prisoner's dilemma game, each player only needs one bit to encode his/her choice: more generally, if there are n pure strategies available for a specific player then $\log_2 n$ bits are needed to specify his/her choice of strategy. Of course, there is nothing forbidding the players from picking their pure strategies randomly. This prompts the mapping of each player's set of strategies to a multi-dimensional simplex. Indeed, the compactness and convexity of the strategy spaces and the multi-linearity of the payoff function are indispensable in proving the Nash Equilibrium Theorem and the Minimax Theorem.

2 Quantum games

Given the many applications mentioned above in which qubits are manipulated rather than classical bits, an immediate approach to quantize classical games would be to replace the bits by qubits. A qubit may be regarded as a quantum system with two states (a so-called two-level system). Physically, it may be represented by the spin of an electron or the ground and excited states of an atom. However this naive approach would not give anything new. A more sophisticated approach was discussed in Ref. [9] and was later shown to be capable of enriching the current scope of classical game theory [17]. Instead of players manufacturing their own qubits, the players in this

approach operate on the qubits sent to them by the referee. Each player then sends his/her manipulated qubit back to the referee. The players' strategy spaces are thus related to the spaces of operators for the qubits. This is similar to the study of quantum error correction: there it is the vector space of error operators on codewords which is of interest, as opposed to the classical case in which it is the vector space of codewords which matters [21]. The full procedure within this game-playing scheme is therefore the following: the referee first sends out sets of qubits to the players; the players then operate separately on the qubits received; they then send the resulting qubits back to the referee, who makes a measurement on these qubits in order to determine the payoffs for each player. As we will show, quantum mechanics plays an essential role in this game by:

1. restricting the feasible set of qubits
2. restricting the physical operations available to the players, and
3. restricting the physical measurements which can be performed by the referee.

For classical games, we already have a clear picture of the meaning of the three restrictions above. Namely, (1) the messages exchanged are always encoded as bit-strings; (2) the operations allowed are restricted to tensor products of the bit-flip (X) and the identity (I) operators, for example $I \otimes X \otimes I(101) = 111$, and randomized ensembles of these operators; (3) the referee just 'reads' the bit-strings received and assigns the payoffs according to the payoff matrix of the game. In contrast, quantum mechanics enforces far less limitation on the vector space of qubits and on the players' strategy spaces. On the other hand, quantum mechanics does not allow perfect state estimation in general and so the referee has limitations on how much he can learn from the qubits. We will now give an axiomatic description on the elements of quantum mechanics that concern us. A more complete treatment may be found in Ref. [21]. We note that all the matrices in this paper can have complex numbers as entries.

Definition 1 (*Description of qubits*) *A set of n qubits is described by a $2^n \times 2^n$ square matrix, ρ , such that*

1. $\text{tr}(\rho)=1$,
2. ρ is a positive matrix.

Any such ρ is called a density matrix.

Definition 2 (*Description of physical operations*) *Given a density matrix ρ , any of the physically-implementable operations on ρ can*

be described by a set of square matrices, $\{E_k\}$, such that the E_k 's are of the same dimension as ρ and $\sum_k E_k^\dagger E_k = I$. Moreover under any physical map $\{E_k\}$, the resulting density matrix will be $\sum_k E_k \rho E_k^\dagger$ which is again a density matrix.

If n qubits are divided into two subsets of n_1 and n_2 qubits, and if each part is operated upon separately, then any physical operation is described by $\{E_k \otimes F_l\}$, where the E_k 's are $(2^{n_1} \times 2^{n_1})$ -matrices and the F_l 's are $(2^{n_2} \times 2^{n_2})$ -matrices such that $\sum_k E_k^\dagger E_k = I$ and $\sum_l F_l^\dagger F_l = I$. The resulting density matrix will be $\sum_{k,l} (E_k \otimes F_l) \rho (E_k^\dagger \otimes F_l^\dagger)$.

Definition 3 (Description of physical measurements) A measurement with L possible outcomes on a density matrix ρ corresponds to L matrices, $\{M_k\}$, of the same dimension as ρ , such that $\sum_{k=1}^L M_k^\dagger M_k = I$. The probability of outcome m is given by $\text{tr}(M_m^\dagger M_m \rho)$.

Given the above definitions, we are now ready to introduce a general theory of quantum games. We will restrict ourselves first to two-player games. We assume that the referee employs the measurement $\{M_k\}$, assigning payoffs a_m^I and a_m^{II} to players I and II respectively if the outcome is m . If players I and II decide to use operations $\{E_k\}$ and $\{F_l\}$ respectively, then the resulting state π will be given by $\sum_{k,l} (E_k \otimes F_l) \rho (E_k^\dagger \otimes F_l^\dagger)$. Hence the payoff for player I is $\sum_{k=1}^L a_k^I \text{tr}(M_k^\dagger M_k \pi) = \text{tr}[(\sum_{k=1}^L a_k^I M_k^\dagger M_k) \pi]$, with a similar expression describing the payoff for player II. It is therefore convenient to denote $\sum_{k=1}^L a_k^I M_k^\dagger M_k$ by R^I , and likewise for R^{II} . We note that R^I, R^{II} and the initial state ρ , define the game completely.

By treating the set of physical maps as a vector space, we may fix a basis for it. We now suppose that $\{\tilde{E}_\alpha\}$ form such a basis. Using $\{E_k = \sum_\alpha e_{k\alpha} \tilde{E}_\alpha\}$ and $\{F_l = \sum_\alpha f_{l\alpha} \tilde{E}_\alpha\}$, then the payoff for player j is

$$\sum_{k,l,\alpha,\beta,\gamma,\delta} e_{k\alpha} \overline{e_{k\beta}} f_{l\gamma} \overline{f_{l\delta}} A_{\alpha\beta\gamma\delta}^j \quad (2)$$

where $A_{\alpha\beta\gamma\delta}^j := \text{tr}[R^j (\tilde{E}_\alpha \otimes \tilde{E}_\gamma) \rho (\tilde{E}_\beta^\dagger \otimes \tilde{E}_\delta^\dagger)]$. Letting $\chi_{\alpha\beta} = \sum_k e_{k\alpha} \overline{e_{k\beta}}$ and $\xi_{\gamma\delta} = \sum_l f_{l\gamma} \overline{f_{l\delta}}$, then χ and ξ are positive Hermitian matrices by construction. In quantum information, the above procedure is called χ matrix representation [21]. We now rewrite Eq. 2 as follows:

$$\sum_{\alpha,\beta,\gamma,\delta} \chi_{\alpha\beta} \xi_{\gamma\delta} A_{\alpha\beta\gamma\delta}^j. \quad (3)$$

We note that the tensors A s fully describes the game being played, and the choices for χ and ξ are limited by the laws of physics. In particular, by denoting the set of allowable χ by Ω , the conditions described in definition 2 are transformed to the following conditions:

1. Ω is a subset of the set of positive Hermitian matrices,
2. For all $\chi \in \Omega$, then $\sum_{\alpha,\beta} \overline{\chi_{\alpha\beta}} \tilde{E}_\alpha^\dagger \tilde{E}_\beta = I$.

We can now see a striking similarity between static quantum games and static classical finite games. The payoff for a classical finite two-player game has the form $\sum_{i,j} x_i A_{ij} y_j$ where x, y belong to some multi-dimensional simplexes and A is a general matrix: the payoff for a static quantum game is $\sum_{\alpha,\beta,\gamma,\delta} \chi_{\alpha\beta} \xi_{\gamma\delta} A_{\alpha\beta\gamma\delta}$ where χ, ξ belong to some multi-dimensional compact and convex sets Ω . Indeed the multi-linear structure of the payoff function together with the convexity and compactness of the strategy sets, are the essential features underlying both classical and quantum games. Indeed, we can exploit these similarities in order to extend some classical results into the quantum domain. Two immediate examples are the Nash Equilibrium Theorem and the Minmax Theorem [17]. We note that the classical strategy set, i.e. a multi-dimensional simplex, and the quantum strategy set Ω cannot be made identical if the linearity of the payoff function is to be preserved. This is because there is no linear homeomorphism that maps Ω_k to a simplex of any dimension. In essence the positivity of Ω_k , i.e. the conditions $\chi_{\alpha\alpha}\chi_{\beta\beta} \geq |\chi_{\alpha\beta}|^2$ for all $\chi \in \Omega_k$, spoils this possibility. Therefore, if we identify Ω_k as some multi-dimensional simplex, we must lose linearity of the payoff function. The structure of the strategy sets in the quantum case therefore introduces new complexity to the study of finite games.

This entire analysis can easily be generalized to N -player games. For instance, any particular N -player static game will have payoff matrices of the form $A^k = \text{tr}[R^k(\tilde{E} \otimes \cdots \otimes \tilde{E})\rho(\tilde{E} \otimes \cdots \otimes \tilde{E})]$ where we have omitted the index summation for clarity.

3 Quantized prisoner's dilemma game

We now apply the formalism developed in the previous section, to discuss the quantized prisoner's dilemma game introduced in Ref. [9]. Considering this game within our formalism, we are led to the following forms for the initial density matrix ρ and the matrices R^I, R^{II} :

$$\rho = \begin{pmatrix} 1/2 & 0 & 0 & -i/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i/2 & 0 & 0 & 1/2 \end{pmatrix}, \quad (4)$$

$$R^I = \begin{pmatrix} 2 & 0 & 0 & -i \\ 0 & 5/2 & 5i/2 & 0 \\ 0 & -5i/2 & 5/2 & 0 \\ i & 0 & 0 & 2 \end{pmatrix} \quad R^{II} = \begin{pmatrix} 2 & 0 & 0 & -i \\ 0 & 5/2 & -5i/2 & 0 \\ 0 & 5i/2 & 5/2 & 0 \\ i & 0 & 0 & 2 \end{pmatrix}. \quad (5)$$

The motivation behind the above forms is as follows: if only the bit-flip and the identity operations are allowed, then the above game reduces to the classical prisoner's dilemma game with the following game matrix [9]:

$$\begin{bmatrix} (3,3) & (0,5) \\ (5,0) & (1,1) \end{bmatrix} \quad (6)$$

To perform concrete calculations, we identify the basis set $\{\tilde{E}_\alpha\}$ as $\{n_{[ij]}\}$ where $n_{[ij]}$ denotes an $n \times n$ square matrix such that the (ij) -entry is equal to 1 while all other entries are equal to zero. Denoting α by ij and recalling the conditions on Ω , we have the following restrictions on all $\chi \in \Omega$: $\sum_i \chi_{ijij} = 1$, $\sum_i \chi_{ijil} = 0$ and $\chi_{ijij}\chi_{klkl} \geq |\chi_{ijkl}|^2$, where the first two sub-indices represent α while the latter two represent β . A further calculation shows that $n_{[ij]} \otimes n_{[kl]} = n_{[(i-1)n+k, (j-1)n+l]}^2$. For arbitrary R and ρ , we find the following:

$$A_{\underbrace{ab}_\alpha \underbrace{cd}_\beta \underbrace{ij}_\gamma \underbrace{kl}_\delta} = R_{((c-1)n+k), ((a-1)n+i) \times \rho_{((b-1)n+j), ((d-1)n+l)}}. \quad (7)$$

We can therefore compute the A 's easily for the above quantized prisoner's dilemma game. These are shown explicitly in Figure 1.

It can be seen that the matrices contain imaginary numbers and negative numbers. However there is no cause for concern: once we properly take the conditions on Ω into account, the resulting payoffs for the two players will always lie between 0 and 5. From the form of the matrices, one can also see that there is a Nash equilibrium with payoff 2.5 for each player: this corresponds to player I adopting the strategy χ^* , where $\chi_{0000}^* = \chi_{0101}^* = 1$ and $\chi^* = 0$ for all other entries and player II adopting the strategy ξ^* , where $\xi_{1010}^* = \xi_{1111}^* = 1$ and $\xi^* = 0$ for all other entries. One can check that the above χ and ξ is contained in Ω , and is hence physically implementable. We note that this is a Nash equilibrium with the highest common payoff which is known in this game.

4 Concluding remarks

We have taken a brief tour through quantum games, and have shown that quantum games are indeed quite distinct from classical games.

Figure 1: The payoff matrices A for the quantized prisoner's dilemma game. The i -th row (column) of A corresponds to the first (last) four sub-indices of A representing the binary expansion of $(i-1)$. For example, A_{65} corresponds to $A_{01010100}$ (c.f. Eq. 7).

Payoff matrix A^I

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5i}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{5i}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{5}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5i}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{5i}{4} & 0 & 0 \\ 0 & 0 & \frac{5i}{4} & 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{5i}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5i}{4} & 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5i}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Payoff matrix A^{II}

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5i}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & \frac{5i}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5i}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{5}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & \frac{5i}{4} & 0 & 0 \\ 0 & 0 & -\frac{5i}{4} & 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{5}{4} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{5i}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5i}{4} & 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{5i}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In particular, the strategy sets are no longer simplexes and the payoff matrices admit complex entries. These new features are certainly interesting from an academic point of view. However the success of classical game theory lies in its applications. To render quantum game theory interesting, one must search for real-life scenarios where quantum games are useful. It turns out that such examples are not hard to find: for example, multi-party communication schemes can naturally be envisaged as a multi-player game. Cryptography is another immediate example [18]. Moreover, game-theoretic language is well-suited to describe scenarios with multi-party interactions: indeed there are many examples of researchers discussing analogies between games and quantum systems long before the words ‘quantum games’ were introduced [19, 23]. In addition, pursuing the underlying concept that information is physical and that physical systems can be seen as information-processors, one is led to the idea that game theory might even provide a novel interpretation of both classical and quantum physics. Such possibilities are likely to ignite future interest in game theory within the physical sciences both from the classical and quantum perspectives. In short, game theory is once again poised to extend its formidable range of application – however, this time the application lies at the heart of fundamental science.

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